Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 258 (2010) 4052–4074

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

A transference theorem for the Dunkl transform and its applications ☆

Feng Dai [a,*], Heping Wang [b]

[a] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada*

[b] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

Received 28 August 2009; accepted 10 March 2010

Available online 29 March 2010

Communicated by N. Kalton

Abstract

For a family of weight functions invariant under a finite reflection group, we show how weighted L^p multiplier theorems for Dunkl transform on the Euclidean space $\mathbb{R}^d$ can be transferred from the corresponding results for h-harmonic expansions on the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$. The result is then applied to establish a Hörmander type multiplier theorem for the Dunkl transform and to show the convergence of the Bochner–Riesz means of the Dunkl transform of order above the critical index in weighted L^p spaces.

Keywords: Hörmander type multiplier theorem; Dunkl transform; Transference theorem; h-Harmonics

1. Introduction

Let R be a reduced root system in $\mathbb{R}^d$ normalized so that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in R$, where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product. Given a nonzero vector $\alpha \in \mathbb{R}^d$, we denote by σ_α the reflection with respect to the hyperplane perpendicular to α; that is, $\sigma_\alpha x = x - 2(\langle x,\alpha\rangle/\|\alpha\|^2)\alpha$ for all $x \in \mathbb{R}^d$, where $\|\cdot\|$ denotes the usual Euclidean norm. Let G denote

☆ The first author was partially supported by the NSERC Canada under grant G121211001. The second author was supported by the National Natural Science Foundation of China (Project No. 10871132) and the Beijing Natural Science Foundation (Project No. 1102011).

* Corresponding author.

E-mail addresses: dfeng@math.ualberta.ca (F. Dai), wanghp@yahoo.cn (H. Wang).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.03.006

the finite subgroup of the orthogonal group $O(d)$ generated by the reflections σ_α, $\alpha \in R$. Let $\kappa : R \to \mathbb{R}_+$ be a nonnegative multiplicity function on R with the property $\kappa(g\alpha) = \kappa(\alpha)$ for all $\alpha \in R$ and $g \in G$. Associated with the reflection group G and the function κ is the weight function h_κ defined by

$$h_\kappa(x) := \prod_{\alpha \in R_+} \left|\langle x, \alpha\rangle\right|^{\kappa(\alpha)}, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where R_+ is an arbitrary but fixed positive subsystem of R. The function h_κ is a homogeneous function of degree $\gamma_\kappa := \sum_{\alpha \in R_+} \kappa(\alpha)$, and is invariant under the reflection group G. For convenience, we shall set $\lambda_\kappa = \frac{d-1}{2} + \gamma_\kappa$ for the rest of the paper. Given $1 \leqslant p \leqslant \infty$, we denote by $L^p(\mathbb{R}^d; h_\kappa^2)$ the weighted Lebesgue space endowed with the norm

$$\|f\|_{\kappa,p} := \left(\int_{\mathbb{R}^d} \left|f(y)\right|^p h_\kappa^2(y)\, dy \right)^{\frac{1}{p}},$$

with the usual change when $p = \infty$.

The Dunkl transform, a generalization of the classical Fourier transform, is defined, for $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, by

$$\mathcal{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $c_\kappa = (\int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\frac{\|x\|^2}{2}}\, dx)^{-1}$, and $E_\kappa(ix, y) = V_\kappa[e^{i\langle x, \cdot\rangle}](y)$ is the weighted analogue of the character $e^{i\langle x, y\rangle}$. Here V_κ is the Dunkl intertwining operator, whose precise definition will be given in Section 2. The Dunkl transform plays the same role as the Fourier transform in classical Fourier analysis, and enjoys properties similar to those of the classical Fourier transform (see [11]).

Several important results in classical Fourier analysis have been extended to the setting of Dunkl transform by Thangavelu and Yuan Xu [18,17]. The problem, however, turns out to be rather difficult in general. One of the difficulties comes from the fact that the generalized translation operator τ_y, which plays the role of the usual translation $f \to f(\cdot - y)$, is not positive in general (see, for instance, [17, Proposition 3.10]). In fact, even the L^p boundedness of τ_y is not established in general (see [18,17]).

In this paper, we shall first prove a transference theorem (Theorem 3.1) between the L^p multiplier of h-harmonic expansions on the unit sphere and that of the Dunkl transform. This theorem, combined with the corresponding results on h-harmonic expansions on the unit sphere recently established in [2–4], is then applied to establish a Hörmander type multiplier theorem for the Dunkl transform (Theorem 4.1), and to show the convergence of the Bochner–Riesz means in the weighted L^p spaces (Theorem 4.3).

The paper is organized as follows. In Section 2, we describe briefly some known results on Dunkl transform and h-harmonic expansions, which will be needed in later sections. The transference theorem, Theorem 3.1, is proved in Section 3. As applications, we prove Theorems 4.1 and 4.3 in the final section, Section 4.

2. Preliminaries

In this section, we shall present some necessary material on the Dunkl transform and the h-harmonic expansions, most of which can be found in [8,11,13,14,17].

2.1. The Dunkl transform

Let R, R_+, G, κ and h_κ be as defined in Section 1. Recall that a reduced root system is a finite subset R of $\mathbb{R}^d \setminus \{0\}$ with the properties $\sigma_\alpha R = R$ and $R \cap \{t\alpha:\ t \in \mathbb{R}\} = \{\pm\alpha\}$ for all $\alpha \in R$. The Dunkl operators associated with G and κ are defined by

$$\mathcal{D}_{\kappa,i} f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} \kappa(\alpha) \frac{f(x) - f(\sigma(\alpha)x)}{\langle x, \alpha \rangle} \langle \alpha, e_i \rangle, \quad 1 \leqslant i \leqslant d, \tag{2.1}$$

where $e_1, \ldots, e_d$ are the standard unit vectors of $\mathbb{R}^d$. Those operators mutually commute, and map $\mathbb{P}_n^d$ to $\mathbb{P}_{n-1}^d$, where $\mathbb{P}_n^d$ is the space of homogeneous polynomials of degree n in d variables (see [5]). We denote by $\Pi^d := \Pi(\mathbb{R}^d)$ the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^d$. An important result in Dunkl theory states that there exists a linear operator $V_\kappa : \Pi^d \to \Pi^d$ determined uniquely by

$$V_\kappa\big(\mathbb{P}_n^d\big) \subset \mathbb{P}_n^d, \qquad V_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_{\kappa,i} V_\kappa = V_\kappa \partial_i, \quad 1 \leqslant i \leqslant d. \tag{2.2}$$

Such an operator is called the intertwining operator. A very useful explicit formula for V_κ was obtained by C.F. Dunkl [6] in the case of $G = \mathbb{Z}_2$, and was later extended to the more general case of $G = \mathbb{Z}_2^d$ ($d \in \mathbb{N}$) by Xuan Xu [21]. In general, one has the following important result of Rösler [13]:

Lemma 2.1. *(See [13, Theorem 1.2 and Corollary 5.3].) For every $x \in \mathbb{R}^d$, there exists a unique probability measure μ_x^κ on the Borel σ-algebra of $\mathbb{R}^d$ such that*

$$V_\kappa P(x) = \int_{\mathbb{R}^d} P(\xi)\, d\mu_x^\kappa(\xi), \quad P \in \Pi^d. \tag{2.3}$$

Furthermore, the representing measures μ_x^κ are compactly supported in the convex hull $C(x) :=$ $\operatorname{co}\{gx:\ g \in G\}$ of the orbit of x under G, and satisfy

$$\mu_{rx}^\kappa(E) = \mu_x^\kappa\big(r^{-1}E\big) \quad \text{and} \quad \mu_{gx}^\kappa(E) = \mu_x^\kappa\big(g^{-1}E\big) \tag{2.4}$$

for all $r > 0$, $g \in G$ and each Borel subset E of $\mathbb{R}^d$.

In particular, the above lemma shows that the intertwining operator V_κ is positive. We point out that Lemma 2.1 will play a crucial role in the analysis of our paper.

By means of (2.3), V_κ can be extended to the space $C(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$. We denote this extension by V_κ again.

The Dunkl transform associated with G and κ is defined by (1.2) with

$$E_\kappa(-ix, y) := V_\kappa\big[e^{-i\langle x, \cdot\rangle}\big](y), \quad x, y \in \mathbb{R}^d. \tag{2.5}$$

If $\kappa = 0$ then $V_\kappa = \mathrm{id}$ and the Dunkl transform coincides with the usual Fourier transform, whereas if $d = 1$ and $G = \mathbb{Z}_2$ then it is closely related to the Hankel transform on the real line.

We list some of the known properties of the Dunkl transform in the following lemma.

Lemma 2.2. *(See [7,11].)*

(i) *If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ then $\mathcal{F}_\kappa f \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\|\to\infty} \mathcal{F}_\kappa f(\xi) = 0$.*
(ii) *The Dunkl transform $\mathcal{F}_\kappa$ is an isomorphism of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ onto itself, and $\mathcal{F}_\kappa^2 f(x) = f(-x)$.*
(iii) *The Dunkl transform $\mathcal{F}_\kappa$ on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2(\mathbb{R}^d; h_\kappa^2)$, i.e., $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.*
(iv) *If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$ then the following inverse formula holds:*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

(v) *If $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ then*

$$\int_{\mathbb{R}^d} \mathcal{F}_\kappa f(x) g(x)\, h_\kappa^2(x)\, dx = \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa g(x)\, h_\kappa^2(x)\, dx.$$

(vi) *Given $\varepsilon > 0$, let $f_\varepsilon(x) = \varepsilon^{-2-2\gamma_\kappa} f(\varepsilon^{-1}x)$. Then $\mathcal{F}_\kappa f_\varepsilon(\xi) = \mathcal{F}_\kappa f(\varepsilon\xi)$.*
(vii) *If $f(x) = f_0(\|x\|)$ is radial, then $\mathcal{F}_\kappa f(\xi) = H_{\lambda_\kappa - \frac{1}{2}} f_0(\|\xi\|)$ is again a radial function, where H_α denotes the Hankel transform defined by*

$$H_\alpha g(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1}\, dr,$$

and J_α denotes the Bessel function of the first kind.

Statements (i)–(vi) of Lemma 2.2 above were proved by M.F.E. de Jeu [11], and were, in fact, contained in Corollaries 4.7 and 4.22, Theorems 4.26 and 4.20, Lemmas 4.13 and 4.3(3) of [11], respectively. Statement (vii) of Lemma 2.2 was proved by Dunkl [7] and was stated more explicitly in [17, Proposition 2.4].

Given $y \in \mathbb{R}^d$, the generalized translation operator $f \to \tau_y f$ is defined on $L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$\mathcal{F}_\kappa(\tau_y f)(\xi) = E_\kappa(-i\xi, y) \mathcal{F}_\kappa f(\xi), \quad \xi \in \mathbb{R}^d.$$

It is known that $\tau_y f(x) = \tau_x f(y)$ for a.e. $x \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$. In general, the operator τ_y is not positive (see, for instance, [17, Proposition 3.10]), and it is still an open problem whether $\tau_y f$ can be extended to a bounded operator on $L^1(\mathbb{R}^d; h_\kappa^2)$. On the other hand, however, it was shown in [17, Theorem 3.7] that the generalized translation operator τ_y can be extended to all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant 2$, and $\tau_y : L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2) \to L^p(\mathbb{R}^d; h_\kappa^2)$ is a bounded operator, where $L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ denotes the space of all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$.

The generalized convolution of $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y)\tau_x \widetilde{g}(y) h_\kappa^2(y)\, dy, \tag{2.6}$$

where $\widetilde{g}(y) = g(-y)$. Since τ_y is a bounded operator from $L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ to $L^p(\mathbb{R}^d; h_\kappa^2)$ for $1 \leqslant p \leqslant 2$, it follows that the definition of $f *_\kappa g$ can be extended to all $g \in L^p_{\mathrm{rad}}(\mathbb{R}^d; h_\kappa^2)$ and $f \in L^{p'}(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p \leqslant 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. The generalized convolution satisfies the following basic property:

$$\mathcal{F}_\kappa(f *_\kappa g)(\xi) = \mathcal{F}_\kappa f(\xi)\mathcal{F}_\kappa g(\xi). \tag{2.7}$$

More properties on the generalized translation operator and the generalized convolution can be found in [17].

2.2. *h-Harmonic expansions*

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d\colon \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual Lebesgue measure $d\sigma(x)$. For the weight function h_κ given in (1.1), we consider the weighted Lebesgue space $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ of functions on $\mathbb{S}^{d-1}$ endowed with the finite norm

$$\|f\|_{L^p(h_\kappa^2;\mathbb{S}^{d-1})} \equiv \|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} \big|f(y)\big|^p h_\kappa^2(y)\, d\sigma(y)\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$ we assume that L^∞ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm $\|f\|_\infty$. We shall use the notation $\|\cdot\|_{\kappa,p}$ to denote the weighted norm for functions defined either on $\mathbb{R}^d$ or on $\mathbb{S}^{d-1}$ whenever it causes no confusion.

A homogeneous polynomial is called an h-harmonic if it is orthogonal to all polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. Let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of all h-harmonics of degree n, and let $\mathrm{proj}_n^\kappa : L^2(h_\kappa^2; \mathbb{S}^{d-1}) \to \mathcal{H}_n^d(h_\kappa^2)$ denote the orthogonal projection operator. The projection proj_n^κ has an integral representation

$$\mathrm{proj}_n^\kappa f(x) := \int_{\mathbb{S}^{d-1}} f(y) P_n^\kappa(x, y) h_\kappa^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{2.8}$$

where $P_n^\kappa(x, y)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$ which can be written in terms of the intertwining operator V_κ as (see [20, Theorem 3.2, (3.1)])

$$P_n^\kappa(x, y) = \frac{n + \lambda_k'}{\lambda_\kappa'} V_\kappa\big[C_n^{\lambda_k'}\big(\langle x, \cdot\rangle\big)\big](y), \quad x, y \in \mathbb{S}^{d-1}, \tag{2.9}$$

with $\lambda_\kappa' := \lambda_\kappa - \frac{1}{2} = \gamma_\kappa + \frac{d-2}{2}$. Here C_n^λ denotes the standard Gegenbauer polynomial of degree n and index λ as defined in [16]. By means of (2.8) and (2.9), the projection $\mathrm{proj}_n^\kappa f$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

The following Marcinkiewicz type multiplier theorem was proved recently in [2, Theorem 2.3]:

Theorem 2.3. *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of real numbers that satisfies*

(i) $\sup_j |\mu_j| \leqslant c < \infty$,
(ii) $\sup_{j\geqslant 1} 2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^r \mu_l| \leqslant c < \infty$, *with r being the smallest integer* $\geqslant \frac{d}{2} + \gamma_\kappa$,

where $\Delta\mu_l = \mu_l - \mu_{l+1}$ and $\Delta^{j+1}\mu_l = \Delta^j \mu_l - \Delta^j \mu_{l+1}$. Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ multiplier for all $1 < p < \infty$; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j^\kappa f \right\|_{\kappa,p} \leqslant A_p c \|f\|_{\kappa,p}, \quad 1 < p < \infty,$$

where A_p is independent of $\{\mu_j\}$ and f.

When $\kappa = 0$, the theorem becomes part (1) of [1, Theorem 4.9] on the ordinary spherical harmonic expansions.

For $\delta > -1$, the Cesàro (C, δ) means of the h-harmonic expansion are defined by

$$S_n^\delta\big(h_\kappa^2; f, x\big) := \big(A_n^\delta\big)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \operatorname{proj}_k^\kappa f(x), \qquad A_{n-k}^\delta = \binom{n-k+\delta}{n-k}.$$

In the case when $G = \mathbb{Z}_2^d$ and $h_\kappa(x) = \prod_{i=1}^d |\langle x, e_i\rangle|^{\kappa(e_i)}$, the following result was proved recently in [3]:

Theorem 2.4. *Let $G = \mathbb{Z}_2^d$ and let $1 \leqslant p \leqslant \infty$ satisfy $|\frac{1}{p} - \frac{1}{2}| \geqslant \frac{1}{2\sigma_\kappa + 2}$ with*

$$\sigma_\kappa := \tfrac{d-2}{2} + \gamma_\kappa - \min_{1\leqslant i\leqslant d} \kappa(e_i).$$

Then

$$\sup_{n\in\mathbb{N}} \big\| S_n^\delta\big(h_\kappa^2; f\big)\big\|_{\kappa,p} \leqslant c\|f\|_{\kappa,p}, \quad \text{for all } f \in L^p\big(h_\kappa^2; \mathbb{S}^{d-1}\big)$$

if and only if

$$\delta > \delta_\kappa(p) := \max\left\{(2\sigma_\kappa + 1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right\}. \tag{2.10}$$

3. A transference theorem

The main goal in this section is to establish a transference theorem between the L^p multipliers of h-harmonic expansions on the unit sphere $\mathbb{S}^d := \{x \in \mathbb{R}^{d+1}\colon \|x\| = 1\}$ and those of the Dunkl transform in $\mathbb{R}^d$. Let G, R, h_κ be as defined in Section 1. Given $g \in G$, we denote by g' the orthogonal transformation on $\mathbb{R}^{d+1}$ determined uniquely by

$$g'x' = (gx, x_{d+1}) \quad \text{for } x' = (x, x_{d+1}) \text{ with } x \in \mathbb{R}^d \text{ and } x_{d+1} \in \mathbb{R}.$$

Then $G' := \{g'\colon g \in G\}$ is a finite reflection group on $\mathbb{R}^{d+1}$ with a reduced root system $R' := \{(\alpha, 0)\colon \alpha \in R\}$. Let κ' denote the nonnegative multiplicity function defined on R' with the property $\kappa'(\alpha, 0) = \kappa(\alpha)$. We denote by $V_{\kappa'}$ the intertwining operator on $C(\mathbb{R}^{d+1})$ associated with the reflection group G' and the multiplicity function κ'. Define the weight function

$$h_{\kappa'}(x, x_{d+1}) := h_\kappa(x) = \prod_{\alpha \in R_+} \bigl|\langle x, \alpha\rangle\bigr|^{\kappa(\alpha)}, \quad x \in \mathbb{R}^d,\ x_{d+1} \in \mathbb{R}.$$

Recall that $\mathrm{proj}_n^{\kappa'} : L^2(\mathbb{S}^d; h_{\kappa'}^2) \to \mathcal{H}_n^{d+1}(h_{\kappa'}^2)$ denotes the orthogonal projection onto the space of h-harmonics.

Our main result is the following.

Theorem 3.1. *Let $m : [0, \infty) \to \mathbb{R}$ be a continuous and bounded function, and let U_ε, $\varepsilon > 0$, be a family of operators on $L^2(\mathbb{S}^d; h_{\kappa'}^2)$ given by*

$$\mathrm{proj}_n^{\kappa'}(U_\varepsilon f) = m(\varepsilon n)\,\mathrm{proj}_n^{\kappa'} f, \quad n = 0, 1, \ldots. \tag{3.1}$$

Assume that

$$\sup_{\varepsilon>0} \|U_\varepsilon f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \leqslant A\|f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)}, \quad \forall f \in C\bigl(\mathbb{S}^d\bigr), \tag{3.2}$$

for some $1 \leqslant p \leqslant \infty$. Then the function $m(\|\cdot\|)$ defines an $L^p(\mathbb{R}^d; h_\kappa^2)$ multiplier; that is,

$$\|T_m f\|_{L^p(\mathbb{R}^d; h_\kappa^2)} \leqslant c_{d,\kappa} A\|f\|_{L^p(\mathbb{R}^d; h_\kappa^2)}, \quad \forall f \in \mathcal{S}\bigl(\mathbb{R}^d\bigr),$$

where T_m is an operator initially defined on $L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$\mathcal{F}_\kappa(T_m f)(\xi) = m\bigl(\|\xi\|\bigr)\mathcal{F}_\kappa f(\xi), \quad f \in L^2\bigl(\mathbb{R}^d; h_\kappa^2\bigr),\ \xi \in \mathbb{R}^d. \tag{3.3}$$

In the case of ordinary spherical harmonics (i.e., $\kappa = 0$), Theorem 3.1 is due to Bonami and Clerc [1, Theorem 1.1].

3.1. Lemmas

The proof of Theorem 3.1 relies on several lemmas.

Lemma 3.2. *If $f \in \Pi^{d+1}$ then for any $x \in \mathbb{R}^d$ and $x_{d+1} \in \mathbb{R}$,*

$$V_{\kappa'} f(x, x_{d+1}) = V_\kappa\big[f(\cdot, x_{d+1})\big](x) = \int\limits_{\mathbb{R}^d} f(\xi, x_{d+1})\, d\mu_x^\kappa(\xi), \tag{3.4}$$

where $d\mu_x^\kappa$ is given in (2.3).

Proof. Clearly, the second equality in (3.4) follows directly from (2.3). To show the first equality, we set $\widetilde{V}_\kappa f(x, x_{d+1}) = V_\kappa[f(\cdot, x_{d+1})](x)$ for $f \in C(\mathbb{R}^{d+1})$ and $x \in \mathbb{R}^d$. Since $V_{\kappa'}$ is a linear operator uniquely determined by (2.2), it suffices to show that the following conditions are satisfied:

$$\widetilde{V}_\kappa\big(\mathbb{P}_n^{d+1}\big) \subset \mathbb{P}_n^{d+1}, \qquad \widetilde{V}_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_{\kappa',i} \widetilde{V}_\kappa = \widetilde{V}_\kappa \partial_i, \quad 1 \leqslant i \leqslant d+1.$$

Indeed, these conditions can be easily verified using the properties of V_κ in (2.2), and the following identities, which follow directly from (2.1):

$$\begin{aligned} \mathcal{D}_{\kappa',i}\, g(x, x_{d+1}) &= \mathcal{D}_{\kappa,i}\big[g(\cdot, x_{d+1})\big](x), \quad 1 \leqslant i \leqslant d, \\ \mathcal{D}_{\kappa',d+1}\, g(x, x_{d+1}) &= \partial_{d+1} g(x, x_{d+1}), \quad \text{for } g \in \Pi^{d+1},\ x \in \mathbb{R}^d \text{ and } x_{d+1} \in \mathbb{R}. \end{aligned}$$

This completes the proof of Lemma 3.2. □

To formulate the next lemma, we define the mapping $\psi : \mathbb{R}^d \to \mathbb{S}^d$ by

$$\psi(x) := \big(\xi \sin\|x\|, \cos\|x\|\big) \quad \text{for } x = \|x\|\xi \in \mathbb{R}^d \text{ and } \xi \in \mathbb{S}^{d-1}.$$

Given $N \geqslant 1$, we denote by $N\mathbb{S}^d := \{x \in \mathbb{R}^{d+1}\colon \|x\| = N\}$ the sphere of radius N in $\mathbb{R}^{d+1}$, and define the mapping $\psi_N : \mathbb{R}^d \to N\mathbb{S}^d$ by

$$\psi_N(x) := N\psi\left(\frac{x}{N}\right) = \left(N\xi \sin\frac{\|x\|}{N}, N\cos\frac{\|x\|}{N}\right) \tag{3.5}$$

with $x = \|x\|\xi \in \mathbb{R}^d$ and $\xi \in \mathbb{S}^{d-1}$.

Lemma 3.3. *If $f : N\mathbb{S}^d \to \mathbb{R}$ is supported in the set $\{x \in N\mathbb{S}^d\colon \arccos(N^{-1}x_{d+1}) \leqslant 1\}$, then*

$$\int\limits_{\mathbb{S}^d} f(Nx) h_{\kappa'}^2(x)\, d\sigma(x) = N^{-2\lambda_\kappa - 1} \int\limits_{B(0,N)} f\big(\psi_N(x)\big) h_\kappa^2(x) \left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx,$$

where $B(0, N) = \{y \in \mathbb{R}^d\colon \|y\| \leqslant N\}$.

Proof. First, using the polar coordinate transformation

$$(\xi, \theta) \in \mathbb{S}^{d-1} \times [0, \pi] \to x := (\xi \sin\theta, \cos\theta) \in \mathbb{S}^d,$$

and the fact that $d\sigma(x) = \sin^{d-1}\theta \, d\theta \, d\sigma(\xi)$, we obtain

$$\begin{aligned}\int_{\mathbb{S}^d} f(Nx) h_{\kappa'}^2(x)\, d\sigma(x) &= \int_0^\pi \left[\int_{\mathbb{S}^{d-1}} f(N\xi \sin\theta, N\cos\theta) h_{\kappa'}^2(\xi\sin\theta, \cos\theta)\, d\sigma(\xi) \right] (\sin\theta)^{d-1}\, d\theta \\ &= \int_0^1 \int_{\mathbb{S}^{d-1}} f(N\xi \sin\theta, N\cos\theta) h_\kappa^2(\theta\xi)\, d\sigma(\xi) \left(\frac{\sin\theta}{\theta}\right)^{d-1+2\gamma_\kappa} \theta^{d-1}\, d\theta,\end{aligned}$$

where the last step uses the identity $h_{\kappa'}(y, y_{d+1}) = h_\kappa(y)$, the fact that h_κ^2 is a homogeneous function of degree $2\gamma_\kappa$, and the assumption that f is supported in the set $\{x \in N\mathbb{S}^d\colon \arccos(N^{-1}x_{d+1}) \leqslant 1\}$. Using the usual spherical coordinate transformation in $\mathbb{R}^d$, the last double integral equals

$$\begin{aligned}&\int_{\|y\|\leqslant 1} f\left(\frac{Ny\sin\|y\|}{\|y\|}, N\cos\|y\|\right) h_\kappa^2(y) \left(\frac{\sin\|y\|}{\|y\|}\right)^{2\lambda_\kappa} dy \\ &\quad = N^{-d-2\gamma_\kappa} \int_{\|x\|\leqslant N} f\left(N\frac{x}{\|x\|}\sin\frac{\|x\|}{N}, N\cos\frac{\|x\|}{N}\right) h_\kappa^2(x) \left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx \\ &\quad = N^{-2\lambda_\kappa - 1} \int_{B(0,N)} f(\psi_N x) h_\kappa^2(x) \left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx,\end{aligned}$$

where the first step uses the homogeneity of the weight h_κ and the change of variables $y = \frac{x}{N}$. This proves the desired formula. □

Remark 3.1. It is easily seen that the restriction $\psi_N|_{B(0,N)}$ of the mapping ψ_N on $B(0, N)$ is a bijection from $B(0, N)$ to $\{x \in N\mathbb{S}^d\colon \arccos(N^{-1}x_{d+1}) \leqslant 1\}$. Thus, given a function $f : B(0, N) \to \mathbb{R}$, there exists a unique function f_N supported in $\{x \in N\mathbb{S}^d\colon \arccos(N^{-1}x_{d+1}) \leqslant 1\}$ such that

$$f_N(\psi_N x) = f(x), \quad \forall x \in B(0, N). \tag{3.6}$$

On the other hand, using Lemma 3.3, we have

$$\int_{\mathbb{S}^d} f_N(Nx) h_{\kappa'}^2(x)\, d\sigma(x) = N^{-2\lambda_\kappa - 1} \int_{B(0,N)} f(x) h_\kappa^2(x) \left(\frac{\sin(\|x\|/N)}{\|x\|/N}\right)^{2\lambda_\kappa} dx. \tag{3.7}$$

The formula (3.7) will play an important role in our proof of Theorem 3.1.

We also need a small observation on a formula of Rösler [14] for $\tau_y f(x)$:

Lemma 3.4. *If $f(x) = f_0(\|x\|)$ is a continuous radial function in $L^2(\mathbb{R}^d; h_\kappa^2)$, then for a.e. $y \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$,*

$$\tau_y f(x) = V_\kappa\left[f_0\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x, \cdot\rangle}\right)\right]\left(\frac{y}{\|y\|}\right). \tag{3.8}$$

Formula (3.8) was first proved in [14] under the assumption that f is a radial Schwartz function. Thangavelu and Yuan Xu [17, Proposition 3.3] later observed that it also holds for radial functions $f \in L(\mathbb{R}^d; h_\kappa^2)$ with $\mathcal{F}_\kappa f \in L(\mathbb{R}^d; h_\kappa^2)$. Clearly, our assumption in Lemma 3.4 is slightly weaker than that of [17, Proposition 3.3].

Lemma 3.4 can be deduced from the result of Rösler [14], using a density argument.

Proof. We first choose a sequence of even, C^∞ functions g_j on $\mathbb{R}$ satisfying

$$\sup_{|t| \leqslant 2^{j+1}} \left|g_j(t) - f_0(t)\right| \leqslant 2^{-j}\left(\int_0^{2^j} s^{2\lambda_\kappa}\, ds\right)^{-\frac{1}{2}}.$$

Let φ_j be an even, C^∞ function on $\mathbb{R}$ such that $\chi_{[2^{-j}, 2^j]}(|t|) \leqslant \varphi_j(t) \leqslant \chi_{[2^{-j-1}, 2^{j+1}]}(|t|)$, and let $f_j(x) \equiv f_{j,0}(\|x\|) := g_j(\|x\|)\varphi_j(\|x\|)$ for $x \in \mathbb{R}^d$. Then it's easily seen that $\{f_j\}$ is a sequence of radial Schwartz functions on $\mathbb{R}^d$ satisfying

$$\lim_{j \to \infty} \sup_{2^{-j} \leqslant |t| \leqslant 2^j} \left|f_{j,0}(t) - f_0(t)\right| = 0 \tag{3.9}$$

and

$$\lim_{j \to \infty} \|f_j - f\|_{\kappa,2} = 0. \tag{3.10}$$

Since each f_j is a radial Schwartz function, by Lemma 2.1 and the already proven case of Lemma 3.4 (see [14]), we obtain

$$\tau_y(f_j)(x) = \int_{\|\xi\| \leqslant 1} f_{j,0}\left(\sqrt{\|x\|^2 + \|y\|^2 - 2\|y\|\langle x, \xi\rangle}\right) d\mu^\kappa_{y/\|y\|}(\xi). \tag{3.11}$$

Next, we fix $y \in \mathbb{R}^d$, and set

$$A_n \equiv A_n(y) := \left\{x \in \mathbb{R}^d\colon\ 2^{-n} \leqslant \left|\|x\| - \|y\|\right| \leqslant \|x\| + \|y\| \leqslant 2^n\right\}$$

for $n \in \mathbb{N}$ and $n \geqslant n_0(y) := [\log\|y\|/\log 2] + 1$. Since

$$\left(\|x\| - \|y\|\right)^2 \leqslant \left|\|x\|^2 + \|y\|^2 - 2\|y\|\langle x, \xi\rangle\right| \leqslant \left(\|x\| + \|y\|\right)^2$$

for all $\|\xi\| \leqslant 1$, it follows by (3.9) that

$$\lim_{j\to\infty} f_{j,0}\bigl(\sqrt{\|x\|^2+\|y\|^2-2\|y\|\langle x,\xi\rangle}\,\bigr) = f_0\bigl(\sqrt{\|x\|^2+\|y\|^2-2\|y\|\langle x,\xi\rangle}\,\bigr)$$

uniformly for $x \in A_n(y)$ and $\|\xi\| \leqslant 1$. This together with (3.11) and Lemma 2.1 implies

$$\begin{aligned}\lim_{j\to\infty} \tau_y(f_j)(x) &= \int\limits_{\|\xi\|\leqslant 1} f_0\bigl(\sqrt{\|x\|^2+\|y\|^2-2\|y\|\langle x,\xi\rangle}\,\bigr)\, d\mu^\kappa_{y/\|y\|}(\xi)\\ &= V_\kappa\bigl[f_0\bigl(\sqrt{\|x\|^2+\|y\|^2-2\|y\|\langle x,\cdot\rangle}\,\bigr)\bigr]\left(\frac{y}{\|y\|}\right)\end{aligned}$$

for every $x \in A_n(y)\setminus\{0\}$ and $n \geqslant n_0(y)$. On the other hand, however, by (3.10), we have

$$\lim_{j\to\infty}\bigl\|\tau_y(f_j)-\tau_y f\bigr\|_{\kappa,2}=0$$

for all $y\in\mathbb{R}^d$. Thus,

$$\tau_y(f)(x) = V_\kappa\bigl[f_0\bigl(\sqrt{\|x\|^2+\|y\|^2-2\|y\|\langle x,\cdot\rangle}\,\bigr)\bigr]\left(\frac{y}{\|y\|}\right)$$

for a.e. $x\in A_n(y)$ and all $n\geqslant n_0(y)$. Finally, observing that the set

$$\mathbb{R}^d\setminus\left(\bigcup_{n=n_0(y)}^{\infty} A_n(y)\right) = \bigl\{x\in\mathbb{R}^d\colon\ \|x\|=\|y\|\bigr\}$$

has measure zero in $\mathbb{R}^d$, we deduce the desired conclusion. □

Remark 3.2. By (2.4) and the supporting condition of the measure $d\mu^\kappa_x$, we observe that

$$V_\kappa F(rx) = \int\limits_{\mathbb{R}^d} F(r\xi)\, d\mu^\kappa_x(\xi), \quad \text{for all } F\in C\bigl(\mathbb{R}^d\bigr),\ x\in\mathbb{R}^d, \text{ and } r>0. \tag{3.12}$$

Thus, (3.8) can be rewritten more symmetrically as

$$\tau_y f(x) = V_\kappa\bigl[f_0\bigl(\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle}\,\bigr)\bigr](y). \tag{3.13}$$

Lemma 3.5. *Let $\Phi\in L^1(\mathbb{R},|x|^{2\lambda_\kappa})$ be an even, bounded function on $\mathbb{R}$, and let T_Φ be an operator $L^2(\mathbb{R}^d;h^2_\kappa)\to L^2(\mathbb{R}^d;h^2_\kappa)$ defined by*

$$\mathcal{F}_\kappa(T_\Phi f)(\xi) := \mathcal{F}_\kappa f(\xi)\Phi\bigl(\|\xi\|\bigr),\quad f\in L^2\bigl(\mathbb{R}^d;h^2_\kappa\bigr).$$

Then T_Φ has an integral representation

$$T_\Phi f(x) = \int\limits_{\mathbb{R}^d} f(y)K(x,y)h^2_\kappa(y)\,dy, \quad \textit{for } f\in\mathcal{S}\bigl(\mathbb{R}^d\bigr) \textit{ and a.e. } x\in\mathbb{R}^d,$$

where

$$K(x,y)=c\int_0^\infty \Phi(s)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac{1}{2}}(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})}{(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})^{\lambda_\kappa-\frac{1}{2}}}\right](y)s^{2\lambda_\kappa}\,ds. \tag{3.14}$$

Furthermore, $K(x,y)=K(y,x)$ *for a.e.* $x\in\mathbb{R}^d$ *and a.e.* $y\in\mathbb{R}^d$.

Proof. Let $g(x)=H_{\lambda_\kappa-\frac{1}{2}}\Phi(\|x\|)$, where $x\in\mathbb{R}^d$ and H_α denotes the Hankel transform. Since Φ is an even function in $L^1(\mathbb{R},|x|^{2\lambda_\kappa})\cap L^\infty(\mathbb{R})$, it follows by the properties of the Hankel transform that g is a continuous radial function in $L^2(\mathbb{R}^d;h_\kappa^2)$ and $\mathcal{F}_\kappa g(\xi)=\Phi(\|\xi\|)$. Thus, using (2.7), we have

$$T_\Phi f(x)=f*_\kappa g(x)=\int_{\mathbb{R}^d} f(y)\tau_y g(x)h_\kappa^2(y)\,dy$$

for $f\in L^2(\mathbb{R}^d;h_\kappa^2)$. Since g is a continuous radial function in $L^2(\mathbb{R}^d;h_\kappa^2)$, by Lemma 3.4 and Remark 3.2 it follows that

$$\begin{aligned}K(x,y):=\tau_y g(x)&=V_\kappa\big[H_{\lambda_\kappa-\frac{1}{2}}\Phi\big(\sqrt{\|x\|^2+\|y\|^2+2\langle x,\cdot\rangle}\big)\big](y)\\&=c\int_0^\infty \Phi(s)V_\kappa\left[\frac{J_{\lambda_\kappa-\frac{1}{2}}(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})}{(s\sqrt{\|x\|^2+\|y\|^2-2\langle x,\cdot\rangle})^{\lambda_\kappa-\frac{1}{2}}}\right](y)s^{2\lambda_\kappa}\,ds,\end{aligned}$$

where the last step uses (2.3), the inequality

$$\left|\Phi(s)\frac{J_{\lambda_\kappa-\frac{1}{2}}(rs)}{(rs)^{\lambda_\kappa-\frac{1}{2}}}\right|\leqslant c\big|\Phi(s)\big|$$

and Fubini's theorem. This proves the desired equation (3.14). That $K(x,y)=K(y,x)$ follows from the fact that $\tau_x g(y)=\tau_y g(x)$. □

Our final lemma is a well-known result for the ultraspherical polynomials:

Lemma 3.6. *(See [16, (8.1.1), p. 192].) For* $z\in\mathbb{C}$ *and* $\mu\geqslant 0$,

$$\lim_{k\to\infty} k^{1-2\mu}C_k^\mu\left(\cos\frac{z}{k}\right)=\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(2\mu)}\left(\frac{z}{2}\right)^{-\mu+\frac{1}{2}}J_{\mu-\frac{1}{2}}(z). \tag{3.15}$$

This formula holds uniformly in every bounded region of the complex z-plane.

3.2. Proof of Theorem 3.1

We follow the idea of the proof of Theorem 1.1 of [1]. We first prove the theorem under the additional assumption $|m(t)| \leqslant c_1 e^{-c_2 t}$ for all $t > 0$ and some $c_1, c_2 > 0$. By Lemma 3.5, the operator T_m has an integral representation

$$T_m f(x) = \int_{\mathbb{R}^d} f(y) K(x, y) h_\kappa^2(y)\, dy,$$

where $K(x, y)$ is given by (3.14) with $\Phi = m$. Thus, it is sufficient to prove that

$$I := \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(x) K(x, y) h_\kappa^2(x) h_\kappa^2(y)\, dx\, dy \right| \leqslant cA \tag{3.16}$$

whenever $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^{p'}(\mathbb{R}^d; h_\kappa^2)$ both have compact supports and satisfy $\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)} = \|g\|_{L^{p'}(\mathbb{R}^d;h_\kappa^2)} = 1$.

To this end, we choose a positive number N to be sufficiently large so that the supports of f and g are both contained in the ball $B(0, N)$. By Remark 3.1, there exist functions f_N and g_N both supported in $\{x \in N\mathbb{S}^d\colon \arccos(N^{-1} x_{d+1}) \leqslant 1\}$ and satisfying

$$f_N\big(\psi_N(x)\big) = f(x), \qquad g_N\big(\psi_N(x)\big) = g(x), \quad x \in \mathbb{R}^d, \tag{3.17}$$

where ψ_N is defined by (3.5). It's easily seen from (3.7) that

$$\big\| f_N(N\cdot)\big\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \leqslant cN^{-\frac{2\lambda_\kappa+1}{p}}, \qquad \big\| g_N(N\cdot)\big\|_{L^{p'}(\mathbb{S}^d; h_{\kappa'}^2)} \leqslant cN^{-\frac{2\lambda_\kappa+1}{p'}}.$$

Thus, using (2.8), (2.9), (3.1) and the assumption (3.2) with $\varepsilon = \frac{1}{N}$, we obtain

$$\begin{aligned} I_N &:= N^{2\lambda_\kappa+1} \left| \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \left[\sum_{n=0}^\infty m\big(N^{-1} n\big) P_n^{\kappa'}(x, y) \right] f_N(Ny) g_N(Nx) h_{\kappa'}^2(x) h_{\kappa'}^2(y)\, d\sigma(x)\, d\sigma(y) \right| \\ &\leqslant cA, \end{aligned} \tag{3.18}$$

where $P_n^{\kappa'}(x, y) = \frac{n+\lambda_\kappa}{\lambda_\kappa} V_{\kappa'}[C_n^{\lambda_\kappa}(\langle x, \cdot\rangle)](y)$. Setting

$$H_N(x, y) = N^{-2\lambda_\kappa-1} \sum_{n=0}^\infty m\big(N^{-1} n\big) P_n^{\kappa'}\left(\psi\left(\frac{x}{N}\right), \psi\left(\frac{y}{N}\right)\right),$$

and invoking (3.17) and Lemma 3.3, we obtain

$$I_N = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} H_N(x,y) f(y) g(x) h_\kappa^2(x) h_\kappa^2(y) \left(\frac{\sin(\|x\|/N)}{\|x\|/N} \right)^{2\lambda_\kappa} \times \left(\frac{\sin(\|y\|/N)}{\|y\|/N} \right)^{2\lambda_\kappa} dx\, dy \right|. \tag{3.19}$$

On the other hand, setting

$$b_N(\rho,x,y) = N^{-2\lambda_\kappa - 1} \sum_{n=0}^{\infty} m\left(\frac{n}{N}\right) P_n^{\kappa'}\left(\psi\left(\frac{x}{N}\right), \psi\left(\frac{y}{N}\right)\right) \left(\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\, dt \right)^{-1} \chi_{[\frac{n}{N}, \frac{n+1}{N})}(\rho),$$

we have

$$H_N(x,y) = \int_0^\infty b_N(\rho,x,y) \rho^{2\lambda_\kappa}\, d\rho.$$

Hence, by (3.19),

$$I_N = \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\int_0^\infty b_N(\rho,x,y) \rho^{2\lambda_\kappa}\, d\rho \right] f(y) g(x) h_\kappa^2(x) h_\kappa^2(y) \times \left(\frac{\sin(\|x\|/N)}{\|x\|/N} \right)^{2\lambda_\kappa} \left(\frac{\sin(\|y\|/N)}{\|y\|/N} \right)^{2\lambda_\kappa} dx\, dy \right|. \tag{3.20}$$

The key ingredient in our proof is to show that $\lim_{N\to\infty} I_N = cI$, where c is a constant depending only on d and κ. In fact, once this is proven, then the desired estimate (3.16) will follow immediately from (3.18).

To show $\lim_{N\to\infty} I_N = cI$, we make the following two assertions:

Assertion 1. For any $N > 0$ and $x, y \in \mathbb{R}^d$,

$$\left| b_N(\rho,x,y) \right| \leqslant c e^{-c_2 \rho},$$

where c is independent of x, y and N.

Assertion 2. For any fixed $x, y \in \mathbb{R}^d$ and $\rho > 0$,

$$\lim_{N\to\infty} b_N(\rho,x,y) = c m(\rho) V_\kappa \left[\frac{J_{\lambda_\kappa - \frac{1}{2}}(\rho u(x,y,\cdot))}{(\rho u(x,y,\cdot))^{\lambda_\kappa - \frac{1}{2}}} \right](y), \tag{3.21}$$

where $u(x,y,\xi) = \sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \xi \rangle}$, and c is a constant depending only on d and κ.

For the moment, we take the above two assertions for granted, and proceed with the proof of Theorem 3.1. By Assertion 1 and Hölder's inequality, we can apply the dominated convergence theorem to the integrals in (3.20), and obtain

$$\lim_{N\to\infty} I_N = \left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\int_0^\infty \lim_{N\to\infty} b_N(\rho,x,y)\rho^{2\lambda_\kappa}\,d\rho \right] f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)\,dx\,dy \right|,$$

which, using Assertion 2, equals

$$= c\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\int_0^\infty m(\rho) V_\kappa \left[\frac{J_{\lambda_\kappa-\frac12}(\rho u(x,y,\cdot))}{(\rho u(x,y,\cdot))^{\lambda_\kappa-\frac12}} \right](y)\rho^{2\lambda_\kappa}\,d\rho \right] f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)\,dx\,dy \right|$$

$$= c\left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y) f(y)g(x)h_\kappa^2(x)h_\kappa^2(y)\,dx\,dy \right| = cI,$$

where the second step uses (3.14). Thus, we have shown the desired relation $\lim_{N\to\infty} I_N = cI$, assuming Assertions 1 and 2.

Now we return to the proofs of Assertions 1 and 2. We start with the proof of Assertion 1. Assume that $\frac{n}{N} \leqslant \rho < \frac{n+1}{N}$ for some $n \in \mathbb{Z}_+$. Then $|m(\frac{n}{N})| \leqslant c_1 e^{-c_2\frac{n}{N}} \leqslant ce^{-c_2\rho}$, and $\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\,dt \geqslant cN^{-1}\rho^{2\lambda_\kappa}$. Hence,

$$\begin{aligned} \left|b_N(\rho,x,y)\right| &= N^{-2\lambda_\kappa-1}\left| m\left(\frac{n}{N}\right) P_n^{\kappa'}\left(\psi\left(\frac{x}{N}\right),\psi\left(\frac{y}{N}\right)\right)\right| \left(\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\,dt\right)^{-1} \\ &\leqslant cN^{-2\lambda_\kappa}\rho^{-2\lambda_\kappa}e^{-c_2\rho}\frac{n+\lambda_\kappa}{\lambda_\kappa}\left| V_{\kappa'}\left[C_n^{\lambda_\kappa}\left(\left\langle \psi\left(\frac{x}{N}\right),\cdot\right\rangle\right)\right]\left(\psi\left(\frac{y}{N}\right)\right)\right| \\ &\leqslant c(N\rho)^{-2\lambda_\kappa}e^{-c_2\rho}n^{2\lambda_\kappa} \leqslant ce^{-c_2\rho}, \end{aligned}$$

where we used (2.9) in the second step, and the positivity of V_κ and the estimate $|C_n^{\lambda_\kappa}(t)| \leqslant cn^{2\lambda_\kappa-1}$ in the third step. This proves Assertion 1.

Next, we show Assertion 2. A straightforward calculation shows that for $\frac{n}{N} \leqslant \rho \leqslant \frac{n+1}{N}$ and $\rho > 0$,

$$\left(\int_{\frac{n}{N}}^{\frac{n+1}{N}} t^{2\lambda_\kappa}\,dt\right)^{-1} = \frac{N}{\rho^{2\lambda_\kappa}}\big(1+o_\rho(1)\big), \quad \text{as } N\to\infty.$$

This implies that for $\frac{n}{N} \leqslant \rho \leqslant \frac{n+1}{N}$ and $\rho > 0$,

$$\begin{aligned} b_N(\rho, x, y) &= m(\rho) \frac{n^{2\lambda_\kappa}}{(N\rho)^{2\lambda_\kappa}} n^{-2\lambda_\kappa} P_n^{\kappa'}\left(\psi\left(\frac{x}{N}\right), \psi\left(\frac{y}{N}\right)\right)\big(1 + o_\rho(1)\big) \\ &= cm(\rho) n^{-2\lambda_\kappa+1} V_{\kappa'}\left[C_n^{\lambda_\kappa}\left(\left\langle \psi\left(\frac{x}{N}\right), \cdot\right\rangle\right)\right]\left(\frac{y}{\|y\|} \sin\frac{\|y\|}{N}, \cos\frac{\|y\|}{N}\right) + o_\rho(1), \end{aligned}$$

where we used the continuity of m in the first step, and the estimate $n^{-2\lambda_\kappa}|P_n^{\kappa'}(\psi(\frac{x}{N}), \psi(\frac{y}{N}))| \leqslant c$, as well as the fact that $\lim_{N\to\infty} \frac{n^{2\lambda_\kappa}}{(N\rho)^{2\lambda_\kappa}} = 1$ in the last step (see [4]). Thus, using Lemma 3.2 and (3.12), we obtain

$$\begin{aligned} b_N(\rho, x, y) &= cm(\rho) n^{-2\lambda_\kappa+1} \int_{\mathbb{R}^d} C_n^{\lambda_\kappa}\left(\frac{1}{\|x\|} \sin\frac{\|x\|}{N} \sum_{j=1}^d x_j \xi_j + \cos\frac{\|y\|}{N} \cos\frac{\|x\|}{N}\right) \\ &\quad \times d\mu^\kappa_{\frac{y}{\|y\|}\sin\frac{\|y\|}{N}}(\xi) + o_\rho(1) \\ &= cm(\rho) n^{-2\lambda_\kappa+1} \int_{\|\xi\| \leqslant \|y\|} C_n^{\lambda_\kappa}\big(\cos\theta_N(x, y, \xi)\big)\, d\mu_y^\kappa(\xi) + o_\rho(1), \end{aligned} \tag{3.22}$$

where $\theta_N(x, y, \xi) \in [0, \pi]$ satisfies

$$\cos\theta_N(x, y, \xi) = \left(\frac{1}{\|x\|\|y\|} \sum_{j=1}^d x_j \xi_j\right) \sin\frac{\|x\|}{N} \sin\frac{\|y\|}{N} + \cos\frac{\|x\|}{N} \cos\frac{\|y\|}{N}.$$

Since

$$\begin{aligned} \cos\theta_N(x, y, \xi) &= 1 - \frac{1}{2N^2}\left(\|x\|^2 + \|y\|^2 - 2\sum_{j=1}^d x_j \xi_j\right) + O_{\|x\|, \|y\|}\big(N^{-4}\big) \\ &= 1 - \frac{1}{2N^2} u(x, y, \xi)^2 + O_{\|x\|, \|y\|}\big(N^{-4}\big), \end{aligned}$$

it follows that

$$\begin{aligned} \theta_N(x, y, \xi) &= 2\arcsin\left(\frac{1}{2N}\sqrt{u(x, y, \xi)^2 + O_{\|x\|, \|y\|}\big(N^{-2}\big)}\right) \\ &= \frac{1}{N}\sqrt{u(x, y, \xi)^2 + O_{\|x\|, \|y\|}\big(N^{-2}\big)} + O_{\|x\|, \|y\|}\big(N^{-2}\big) \\ &= \frac{\rho u(x, y, \xi) + o_{\|x\|, \|y\|, \rho}(1)}{n}, \end{aligned}$$

where the last step uses the uniform continuity of the function $t \in [0, M] \to \sqrt{t}$ for any $M > 0$, and the relation $\lim_{N\to\infty} \frac{n}{N\rho} = 1$.

Thus, by (3.22) and (3.15), we have

$$\lim_{N\to\infty} b_N(\rho,x,y) = cm(\rho) \lim_{N\to\infty} \int\limits_{\|\xi\|\leqslant\|y\|} n^{-2\lambda_\kappa+1} C_n^{\lambda_\kappa}\left(\cos\frac{\rho u(x,y,\xi)+o_{x,y,\rho}(1)}{n}\right) d\mu_y^\kappa(\xi)$$

$$= cm(\rho) \int\limits_{\|\xi\|\leqslant\|y\|} \bigl(\rho u(x,y,\xi)\bigr)^{-\lambda_\kappa+\frac12} J_{\lambda_\kappa-\frac12}\bigl(\rho u(x,y,\xi)\bigr)\, d\mu_y^\kappa(\xi)$$

$$= cm(\rho) V_\kappa\bigl[\bigl(\rho u(x,y,\cdot)\bigr)^{-\lambda_\kappa+\frac12} J_{\lambda_\kappa-\frac12}\bigl(\rho u(x,y,\cdot)\bigr)\bigr](y),$$

where we used the fact that $\|C_n^{\lambda_\kappa}\|_\infty \leqslant cn^{2\lambda_\kappa-1}$, the bounded convergence theorem and (3.15) in the last step. This proves Assertion 2.

In summary, we have shown the theorem with the additional assumption $|m(t)| \leqslant c_1 e^{-c_2 t}$.

Finally, we prove that the conclusion of Theorem 3.1 remains true without the additional assumption $|m(t)| \leqslant c_1 e^{-c_2 t}$. To this end, let $m_\delta(t) = m(t)e^{-\delta t}$ for $\delta > 0$, and define T_{m_δ}: $L^2(\mathbb{R}^d, h_\kappa^2) \to L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$\mathcal{F}_\kappa(T_{m_\delta} f)(\xi) = m_\delta(\xi)\mathcal{F}_\kappa f(\xi), \quad f \in L^2\bigl(\mathbb{R}^d; h_\kappa^2\bigr).$$

It is known (see [8, p. 191]) that given any $\varepsilon > 0$, $f \mapsto \sum_{n=0}^\infty e^{-n\varepsilon} \operatorname{proj}_n^{\widetilde{\kappa}} f$ is a positive operator on $L^p(\mathbb{S}^d; h_{\widetilde{\kappa}}^2)$ that satisfies

$$\sup_{\varepsilon>0}\left\| \sum_{n=0}^\infty e^{-n\varepsilon} \operatorname{proj}_n^{\widetilde{\kappa}} f \right\|_{L^p(\mathbb{S}^d; h_{\widetilde{\kappa}}^2)} \leqslant \|f\|_{L^p(\mathbb{S}^d; h_{\widetilde{\kappa}}^2)}.$$

Indeed, this follows from [6, Theorem 4.2] and the fact that V_κ is positive, which was proved in [13]. Thus, applying Theorem 3.1 for the already proven case, we have

$$\sup_{\delta>0} \|T_{m_\delta} f\|_{L^p(\mathbb{R}^d; h_\kappa^2)} \leqslant cA\|f\|_{L^p(\mathbb{R}^d; h_\kappa^2)}. \tag{3.23}$$

On the other hand, from the definition we can decompose the operator T_{m_δ} as

$$T_{m_\delta} f = P_\delta(Tf), \tag{3.24}$$

where $\mathcal{F}_\kappa(Tf)(\xi) = m(\|\xi\|)\mathcal{F}_\kappa f(\xi)$ and

$$\mathcal{F}_\kappa(P_\delta f)(\xi) = e^{-\delta\|\xi\|}\mathcal{F}_\kappa f(\xi).$$

The function $P_\delta f$ is called the Poisson integral of f, and it can be expressed as a generalized convolution (see [3])

$$P_\delta f(x) := (f *_\kappa P_\delta)(x)$$

with

$$P_\delta(x) := 2^{\gamma_\kappa+\frac{d}{2}} \frac{\Gamma(\gamma_\kappa + \frac{d+1}{2})}{\sqrt{\pi}} \frac{\delta}{(\delta^2 + \|x\|^2)^{\gamma_\kappa+\frac{d+1}{2}}}.$$

It was shown in [3, Theorem 6.2] that

$$\lim_{\delta\to 0+} P_\delta f(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d$$

for any $f \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant q < \infty$. Since m is bounded, $Tf \in L^2(\mathbb{R}^d; h_\kappa^2)$ for $f \in L^2(\mathbb{R}^d; h_\kappa^2)$. Thus, for any $f \in \mathcal{S}$, using (3.24),

$$\lim_{\delta\to 0+} T_{m_\delta} f(x) = \lim_{\delta\to 0+} P_\delta(Tf)(x) = Tf(x), \quad \text{a.e. } x \in \mathbb{R}^d, \tag{3.25}$$

which combined with (3.23) and the Fatou theorem implies the desired estimate

$$\|Tf\|_{L^p(\mathbb{R}^d;h_\kappa^2)} \leqslant cA\|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

This completes the proof of the theorem.

4. Applications

4.1. Hörmander's multiplier theorem and the Littlewood–Paley inequality

As a first application of Theorem 3.1, we shall prove the following Hörmander type multiplier theorem:

Theorem 4.1. *Let $m : (0,\infty) \to \mathbb{R}$ be a bounded function satisfying $\|m\|_\infty \leqslant A$ and Hörmander's condition*

$$\frac{1}{R}\int_R^{2R} \left|m^{(r)}(t)\right| dt \leqslant AR^{-r}, \quad \textit{for all } R > 0, \tag{4.1}$$

where r is the smallest integer $\geqslant \lambda_\kappa + 1$. Let T_m be an operator on $L^2(\mathbb{R}^d; h_\kappa^2)$ defined by

$$\mathcal{F}_\kappa(T_m f)(\xi) = m\big(\|\xi\|\big)\mathcal{F}_\kappa f(\xi), \quad \xi \in \mathbb{R}^d.$$

Then

$$\|T_m f\|_{\kappa,p} \leqslant C_p A\|f\|_{\kappa,p}$$

for all $1 < p < \infty$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

Proof. Let $\mu_\ell = m(\ell\varepsilon)$ for $\varepsilon > 0$ and $\ell = 0, 1, \dots$. Then

$$\begin{aligned} \left|\Delta^r \mu_\ell\right| &= \varepsilon^r \left| \int\limits_{[0,1]^r} m^{(r)}(\varepsilon t_1 + \cdots + \varepsilon t_r + \varepsilon\ell)\, dt_1 \cdots dt_r \right| \\ &\leqslant \int\limits_{[0,\varepsilon]^r} \left|m^{(r)}(t_1 + \cdots + t_r + \varepsilon\ell)\right| dt_1 \cdots dt_r \leqslant \varepsilon^{r-1} \int\limits_{\varepsilon\ell}^{\varepsilon(r+\ell)} \left|m^{(r)}(t)\right| dt. \end{aligned}$$

This implies that for $2^j \geqslant r$,

$$\begin{aligned} 2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} \left|\Delta^r \mu_l\right| &\leqslant 2^{j(r-1)} \varepsilon^{r-1} \sum_{l=2^j}^{2^{j+1}} \int\limits_{\varepsilon\ell}^{\varepsilon(r+\ell)} \left|m^{(r)}(t)\right| dt \\ &\leqslant (r-1) 2^{j(r-1)} \varepsilon^{r-1} \int\limits_{2^j\varepsilon}^{\varepsilon(2^{j+1}+r)} \left|m^{(r)}(t)\right| dt \\ &\leqslant 2^{j(r-1)} (r-1) \varepsilon^{r-1} \int\limits_{2^j\varepsilon}^{2^{j+2}\varepsilon} \left|m^{(r)}(t)\right| dt \leqslant c_r A, \end{aligned}$$

where the last step uses (4.1). On the other hand, however, for $2^j \leqslant r$, we have

$$2^{j(r-1)} \sum_{l=2^j}^{2^{j+1}} \left|\Delta^r \mu_l\right| \leqslant c_r \max_j |\mu_j| \leqslant c_r A.$$

Thus, using Theorem 2.3, we deduce

$$\sup_{\varepsilon>0} \left\| \sum_{n=0}^{\infty} m(\varepsilon n) \operatorname{proj}_n^{\kappa'} f \right\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \leqslant c \|f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)}.$$

The desired conclusion then follows by Theorem 3.1. $\square$

Remark 4.1. Hörmander's condition is normally stated in the following form

$$\left(\frac{1}{R} \int\limits_R^{2R} \left|m^{(r)}(t)\right|^2 dt \right)^{\frac{1}{2}} \leqslant A R^{-r}, \quad \text{for all } R > 0. \tag{4.2}$$

See, for instance, [10, Theorem 5.2.7]. Clearly, the condition (4.1) in Theorem 4.1 is weaker than (4.2). On the other hand, however, Theorem 4.1 is applicable only to radial multiplier $m(\|\cdot\|)$.

Corollary 4.2. *Let Φ be an even C^∞-function that is supported in the set $\{x \in \mathbb{R}\colon \frac{9}{10} \leqslant |x| \leqslant \frac{21}{10}\}$ and satisfies either*

$$\sum_{j\in\mathbb{Z}} \Phi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbb{R}\setminus\{0\},$$

or

$$\sum_{j\in\mathbb{Z}} \left|\Phi\left(2^{-j}\xi\right)\right|^2 = 1, \quad \xi \in \mathbb{R}\setminus\{0\}.$$

Let Δ_j be an operator defined by

$$\mathcal{F}_\kappa(\Delta_j f)(\xi) = \Phi\left(2^{-j}\|\xi\|\right)\mathcal{F}_\kappa f(\xi), \quad \xi \in \mathbb{R}^d.$$

Then we have

$$\|f\|_{\kappa,p} \sim_{\kappa,p} \left\| \left(\sum_{j\in\mathbb{Z}} |\Delta_j f|^2 \right)^{\frac{1}{2}} \right\|_{\kappa,p}$$

holds for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $1 < p < \infty$.

Proof. Corollary 4.2 follows directly from Theorem 4.1. Since the proof is quite standard (see, for instance, [15]), we omit the details. □

4.2. The Bochner–Riesz means

Given $\delta > -1$, the Bochner–Riesz means of order δ for the Dunkl transform are defined by

$$B_R^\delta\left(h_\kappa^2; f\right)(x) = c \int\limits_{\|y\|\leqslant R} \left(1 - \frac{\|y\|^2}{R^2}\right)^\delta \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad R > 0. \tag{4.3}$$

Convergence of the Bochner–Riesz means in the setting of Dunkl transform was studied recently by Thangavelu and Yuan Xu [17, Theorem 5.5], who proved that if $\delta > \lambda_\kappa := \frac{d-1}{2} + \gamma_\kappa$ and $1 \leqslant p \leqslant \infty$ then

$$\sup_{R>0} \left\| B_R^\delta\left(h_\kappa^2; f\right)\right\|_{\kappa,p} \leqslant c\|f\|_{\kappa,p}. \tag{4.4}$$

Our next result concerns the critical indices for the validity of (4.4) in the case of $G = \mathbb{Z}_2^d$:

Theorem 4.3. *Suppose that $G = \mathbb{Z}_2^d$, $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$, and $|\frac{1}{p} - \frac{1}{2}| \geqslant \frac{1}{2\lambda_\kappa+2}$. Then* (4.4) *holds if and only if*

$$\delta > \delta_\kappa(p) := \max\left\{ (2\lambda_\kappa + 1)\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0 \right\}. \tag{4.5}$$

It should be pointed out that the result of [17, Theorem 5.5] is applicable to the case of a general finite reflection group G, while Theorem 4.3 above applies to the case of $\mathbb{Z}_2^d$ only.

Proof. We start with the proof of the sufficiency. Assume that $\kappa := (\kappa_1, \dots, \kappa_d)$ and $h_\kappa(x) := \prod_{j=1}^d |x_j|^{\kappa_j}$. Let $\kappa' = (\kappa, 0)$ and $h_{\kappa'}(x, x_{d+1}) = h_\kappa(x)$ for $x \in \mathbb{R}^d$ and $x_{d+1} \in \mathbb{R}$. Set $m(t) = (1 - t^2)_+^\delta$. By the equivalence of the Riesz and the Cesàro summability methods of order $\delta \geqslant 0$ (see [9]), we deduce from Theorem 2.4

$$\sup_{\varepsilon>0} \left\| \sum_{n=0}^{\infty} m(\varepsilon n) \operatorname{proj}_n^{\kappa'} f \right\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)} \leqslant c \|f\|_{L^p(\mathbb{S}^d; h_{\kappa'}^2)}$$

whenever $|\frac{1}{p} - \frac{1}{2}| \geqslant \frac{1}{2\sigma_{\kappa'}+2}$ and $\delta > \delta_{\kappa'}(p)$, where $\sigma_{\kappa'} = \lambda_\kappa$ and $\delta_{\kappa'}(p) = \delta_\kappa(p)$. Thus, invoking Theorem 3.1, we conclude that for $\delta > \delta_\kappa(p)$,

$$\left\| B_1^\delta\big(h_\kappa^2; f\big) \right\|_{\kappa,p} \leqslant c \|f\|_{\kappa,p}.$$

The estimate (4.4) then follows by dilation. This proves the sufficiency.

The necessity part of the theorem follows from the corresponding result for the Hankel transform. To see this, let $f(x) = f_0(\|x\|)$ be a radial function in $L^p(\mathbb{R}^d, h_\kappa^2)$. Using (4.3) and Lemma 2.2(vii), we have (vii), we have

$$B_R^\delta\big(h_\kappa^2; f\big)(x) = \int_0^R \left(1 - \frac{r^2}{R^2}\right)^\delta H_{\lambda_\kappa - \frac{1}{2}} f_0(r) r^{2\lambda_\kappa} \left[\int_{\mathbb{S}^{d-1}} E_\kappa\big(ix, ry'\big) h_\kappa^2\big(y'\big)\, d\sigma\big(y'\big) \right] dr.$$

However, by [17, Proposition 2.3] applied to $n = 0$ and $g = 1$, we have

$$\int_{\mathbb{S}^{d-1}} E_\kappa\big(ix, ry'\big) h_\kappa^2\big(y'\big)\, d\sigma\big(y'\big) = c \left(\frac{r\|x\|}{2}\right)^{-\lambda_\kappa + \frac{1}{2}} J_{\lambda_\kappa - \frac{1}{2}}\big(r\|x\|\big).$$

It follows that

$$\begin{aligned} B_R^\delta\big(h_\kappa^2; f\big)(x) &= c \int_0^R \left(1 - \frac{r^2}{R^2}\right)^\delta H_{\lambda_\kappa - \frac{1}{2}} f_0(r) \left(\frac{r\|x\|}{2}\right)^{-\lambda_\kappa + \frac{1}{2}} J_{\lambda_\kappa - \frac{1}{2}}\big(r\|x\|\big) r^{2\lambda_\kappa}\, dr \\ &= c \widetilde{B}_R^\delta f_0\big(\|x\|\big), \end{aligned}$$

where $\widetilde{B}_R^\delta$ denotes the Bockner–Riesz mean of order δ for the Hankel transform $H_{\lambda_\kappa - \frac{1}{2}}$. However, it is known (see [19]) that $\widetilde{B}_R^\delta$, $0 < \delta < \lambda_\kappa$, is bounded on $L^p((0, \infty), t^{2\lambda_\kappa})$ if and only if

$$\frac{2\lambda_\kappa + 1}{\lambda_\kappa + \delta + 1} < p < \frac{2\lambda_\kappa + 1}{\lambda_\kappa - \delta}. \tag{4.6}$$

Thus, to complete the proof of the necessity part of the theorem, by (4.6), we just need to observe that if $f(x) = f_0(\|x\|)$ is a radial function in $L^p(\mathbb{R}^d; h_\kappa^2)$, then

$$\|f\|_{\kappa,p} = c\|f_0\|_{L^p(\mathbb{R};|x|^{2\lambda_\kappa})}. \qquad \square$$

Remark 4.2. In the unweighted case, for the classical Fourier transform, Theorem 4.3 is well known, and in fact, it follows from the following Tomas–Stein restriction theorem (see, for instance, [10, Section 10.4]):

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1})} \leqslant C_p\|f\|_{L^p(\mathbb{R}^d)}, \quad 1 \leqslant p \leqslant \frac{2d+2}{d+3}, \tag{4.7}$$

where $\widehat{f}$ denotes the usual Fourier transform of f. In the weighted case, while estimates similar to (4.7) can be proved for the Dunkl transform $\mathcal{F}_\kappa f$ (see [12, Theorem 4.1]), they do not seem to be enough for the proof of Theorem 4.3. A similar fact was indicated in [3] for the case of the Cesàro means for h-harmonic expansions on the unit sphere, where global estimates for the projection operators have to be replaced with more delicate local estimates, which are significantly more difficult to prove than the corresponding global estimates.

Acknowledgments

The authors would like to thank the referee for many valuable comments and suggestions, which help to improve the current version of our paper.

References

[1] A. Bonami, J.-L. Clerc, Sommes de Cesàro et multiplicateurs des développe-ments en harmoniques sphériques, Trans. Amer. Math. Soc. 183 (1973) 223–263.
[2] Feng Dai, Yuan Xu, Maximal function and multiplier theorem for weighted space on the unit sphere, J. Funct. Anal. 249 (2) (2007) 477–504.
[3] Feng Dai, Yuan Xu, Boundedness of projection operators and Cesàro means in weighted L^p space on the unit sphere, Trans. Amer. Math. Soc. 361 (6) (2009) 3189–3221.
[4] Feng Dai, Yuan Xu, Cesàro means of orthogonal expansions in several variables, Constr. Approx. 29 (1) (2009) 129–155.
[5] C.F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. 311 (1989) 167–183.
[6] C.F. Dunkl, Integral kernels with reflection group invariance, Canad. J. Math. 43 (6) (1991) 1213–1227.
[7] C.F. Dunkl, Hankel transforms associated to finite reflection groups, in: Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications, Tampa, FL, 1991, in: Contemp. Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 123–138.
[8] C.F. Dunkl, Yuan Xu, Orthogonal Polynomials of Several Variables, Cambridge Univ. Press, 2001.
[9] J.J. Gergen, Summability of double Fourier series, Duke Math. J. 3 (2) (1937) 133–148.
[10] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Inc., Upper Saddle River, NJ, 2004.
[11] M.F.E. de Jeu, The Dunkl transform, Invent. Math. 113 (1993) 147–162.
[12] L.M. Liu, Harmonic analysis associated with finite reflection groups, PhD thesis, 2004, Capital Normal University, Beijing, China.
[13] M. Rösler, Positivity of Dunkl intertwining operator, Duke Math. J. 98 (1999) 445–463.
[14] M. Rösler, A positive radial product formula for the Dunkl kernel, Trans. Amer. Math. Soc. 355 (2003) 2413–2438.
[15] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Math. Ser., vol. 30, Princeton University Press, Princeton, NJ, 1970.
[16] G. Szegö, Orthogonal Polynomials, 4th edition, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[17] S. Thangavelu, Yuan Xu, Convolution operator and maximal function for the Dunkl transform, J. Anal. Math. 97 (2005) 25–55.
[18] S. Thangavelu, Yuan Xu, Riesz transform and Riesz potentials for Dunkl transform, J. Comput. Appl. Math. 199 (1) (2007) 181–195.
[19] G.V. Welland, Norm convergence of Riesz–Bochner means for radial functions, Canad. J. Math. 27 (1) (1975) 176–185.
[20] Yuan Xu, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, Proc. Amer. Math. Soc. 125 (1997) 2963–2973.
[21] Yuan Xu, Orthogonal polynomials for a family of product weight functions on the spheres, Canad. J. Math. 49 (1997) 175–192.